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A GEOMETRICAL PROBLEM CONNECTED WITH THE CONTINUATION OF A POWER-SERIES

BY H. MASCHKE

GIVEN a power-series $P(x)$ with only one singular point A on the circumference of its circle of convergence.

Denote this circle by C_1 , its center by M .

Let us suppose that all the other singular points of the analytic function defined by the element $P(x)$ are so situated as not to interfere with the continuations of $P(x)$ which are to be considered.

The following purely geometrical question arises: *How are the intermediate circles to be chosen in order to arrive again at a circle with center M , by a minimum number of continuations around the point A ?*

To answer this question we consider the total area covered by all direct (first) continuations of $P(x)$. The boundary of this area is the envelope of the circles passing through A whose centers lie on the circumference of C_1 . This envelope is a cardioid C_2 . The boundary of the area covered by all the second continuations is the envelope C_3 of all circles through A whose centers lie on C_2 . Continuing this process, we obtain a series of curves C_2, C_3, \dots, C_n , where C_n is the boundary of the area covered by all $(n-1)$ st continuations.

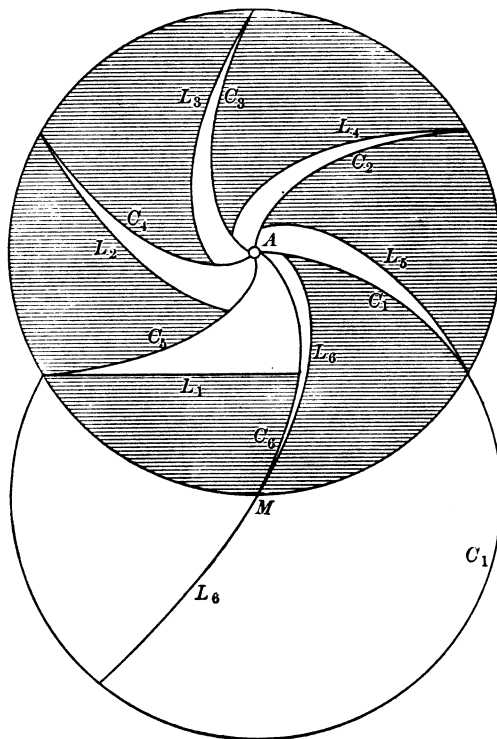
Counting radii vectores ρ from A and polar angles θ from AM , and taking the radius $AM = 1$, the equations of the successive curves appear in

the following simple form :

$$C_1 \text{ (given circle) : } \rho = 2 \cos \theta ;$$

$$C_2 \text{ (cardioid) : } \rho = 4 \cos^2(\tfrac{1}{2} \theta) ;$$

$$C_n \text{ (sine-spiral) : } \rho = 2^n \cos^n(\tfrac{1}{n} \theta) .$$



These curves are so-called sine-spirals.* With regard to the curves C_n the following theorem can easily be proved :

Define a series of points $P_1, P_2, \dots, P_n, \dots$, where P_1 lies arbitrarily on C_1 , and where every P_{n+1} is that point on C_{n+1} (for $n = 1, 2, 3, \dots$)

* Compare Scheffers' article in the *Encyklopädie der mathematischen Wissenschaften*, vol. III, D 4, §§21-24.

in which C_{n+1} is touched by the circle with center P_n and radius $P_n A$. Then for the polar coordinates ρ_n and θ_n of P_n the following relations hold :

$$\rho_n = \rho_1^n, \quad \theta_n = n\theta_1.$$

Choosing $\rho_1 = 1$, we have $\theta_1 = \pi/3$ and every $\rho_n = 1$. Hence, a circle with radius 1 about A meets C_1, C_2, \dots, C_6 in six points which form a regular hexagon. The curve C_6 passes, therefore, through M .

Six continuations at least are then required in order that the center M shall lie inside of the last one, and seven continuations at least are necessary to reach a circle with M as center.

In order that the sixth continuation shall contain the center M it is necessary that the centers of two consecutive continuations be always separated by one of the curves C_n .

In the adjoining figure the curves C_n are drawn only so far as is required for continuations around A in the positive sense (counter-clockwise), and only that part of them is shown in the figure which lies in a circle with radius 1 about the singular point A as center.

Denote now the center of the n^{th} continuation by T_n . If we wish T_7 to coincide with M , T_6 must lie below the straight line L_1 bisecting AM and perpendicular to it (see diagram). This defines for T_5 also a certain limiting curve L_2 , the locus of the centers of circles passing through A and touching L_1 . Likewise we obtain for T_4, T_3, T_2, T_1 the limiting curves L_3, L_4, L_5, L_6 respectively.

The equations of these curves are :

$$\begin{aligned} L_1 \text{ (straight line)} : \quad \frac{1}{\rho} &= 2 \cos \theta ; \\ L_2 \text{ (parabola)} : \quad \frac{1}{\rho} &= 4 \cos^2(\tfrac{1}{2} \theta) ; \\ L_n \text{ (sine-spiral)} : \quad \frac{1}{\rho} &= 2^n \cos^n(\tfrac{1}{n} \theta). \end{aligned}$$

These curves L_n are again sine-spirals.

Every L_n (for $n = 1, 2, \dots, 6$) touches the corresponding C_{6-n} (where $C_0 = C_6$) on the unit-circle around A .

Hence, in our diagram, for continuations around A in the positive sense, none of the centers T must lie in the unshaded region of the circle with center A .

Thus we have reached the following result :

In order that the center T_7 of the seventh continuation shall coincide with M it is necessary and sufficient that the center of the first continuation T_1 (for continuations around A in the positive sense) lie in the circle C_1 on the right side of L_6 (compare diagram), and that the centers T_n of any two consecutive continuations be separated not only by a curve C , but also by a curve L .

UNIVERSITY OF CHICAGO,
OCTOBER, 1905.